# ON SOME CLASS OF MULTIDIMENSIONAL NONLINEAR INTEGRABLE SYSTEMS[a]

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On the base of Lie algebraic and differential geometry methods, a wide class of multidimensional nonlinear integrable systems is obtained, and the integration scheme for such equations is proposed.

**1.** In the report we give a Lie algebraic and differential geometry derivation of a wide class of nonlinear integrable systems of partial differential equations for the functions depending on an arbitrary number of variables, and construct, following the lines of Refs. 1–3, their general solutions in a 'holomorphically factorisable' form. The systems are generated by flat connections, constrained by the relevant grading condition, with values in an arbitrary reductive complex Lie algebra $\mathcal{G}$ endowed with a $\mathbf{Z}$–gradation. They describe a multidimensional version of Toda type fields coupled to matter fields, and, analogously to the two dimensional situation, with an appropriate Inönü–Wigner contraction procedure, for our systems one can exclude back reaction of the matter fields on the Toda fields.

For two dimensional case and the connection taking values in the local part of a finite dimensional algebra $\mathcal{G}$, our equations describe an (abelian and nonabelian) conformal Toda system and its affine deformations for an affine $\mathcal{G}$, see Ref. 1 and references therein, and also Ref. 2 for differential and algebraic geometry background of such systems. For the connection with values in higher grading subspaces of $\mathcal{G}$ one deals with systems discussed in Refs. 3, 4. In higher dimensions our systems, under some additional specialisations, contain as particular cases the Cecotti–Vafa type equations[5] written there for a case of the complexified orthogonal algebra, see also Ref. 6; and those of Gervais–Matsuo[7] which represent some reduction of a generalised WZNW model. Due to the lack of space we present here only an announcement of the results which will be described in detail, together with some remarkable examples elsewhere.

**2.** Let $M$ be the manifold $\mathbf{R}^{2A}$ with the standard coordinates $z^{\pm i}, 1 \leq i \leq A$, or $\mathbf{C}^A$ with $z^{+i} = \overline{z^{-i}}$; let $G$ be a reductive complex Lie group with the Lie algebra $\mathcal{G}$ endowed with a $\mathbf{Z}$–gradation, $\mathcal{G} = \oplus_{m \in \mathbf{Z}} \mathcal{G}_m$. Consider a flat connection $\omega = \sum_{i=1}^{A} (\omega_{-i} dz^{-i} + \omega_{+i} dz^{+i})$ on the trivial principal fiber bundle

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$M \times G \to M$, and impose on it the grading condition that the components $\omega_{\pm i}$ take values in $\mathcal{G}_0 \oplus \mathcal{N}_{\pm i}$, where $\mathcal{N}_{\pm_i} = \bigoplus_{1 \le m \le l_{\pm i}} \mathcal{G}_{\pm m}$ with some positive integers $l_{\pm i}$, such that the subspaces $\mathcal{G}_{\pm l_{\pm i}}$ are nontrivial. Restrict also the connection components $\omega_{\pm i} = \sum_{m=0}^{\pm l_{\pm i}} \omega_{\pm i,m}$ by the condition $\omega_{\pm i, \pm l_{\pm i}} = \zeta_\pm c_{\pm i} \zeta_\pm^{-1}$, where $\zeta_\pm$ are some mappings $M \to G_0$ with $G_0$ being the Lie group corresponding to $\mathcal{G}_0$, and $c_{\pm i}$ are some fixed elements of the subspaces $\mathcal{G}_{\pm l_{\pm i}}$ satisfying the relations $[c_{\pm i}, c_{\pm j}] = 0$. Then one can prove the following statement.

*There exists a local $G_0$–gauge transformation that brings a connection satisfying the above given conditions to the connection $\omega$ with the components*

$$\omega_{+i} = \gamma^{-1} \left( \sum_{m=1}^{l_{+i}-1} \upsilon_{+i,m} + c_{+i} \right) \gamma,$$

$$\omega_{-i} = \gamma^{-1} \partial_{-i} \gamma + \sum_{m=-1}^{-l_{-i}+1} \upsilon_{-i,m} + c_{-i},$$

*where $\gamma$ is some mapping from $M$ to $G_0$, and $\upsilon_{\pm i,m}$ are mappings taking values in $\mathcal{G}_{\pm m}$.*

The equations for the mappings $\gamma$ and $\upsilon_{\pm i,m}$ which follow from the flatness condition we call multidimensional Toda type systems, and the corresponding functions parametrising the mappings $\gamma$ and $\upsilon_{\pm i,m}$ — Toda and matter type fields, respectively. In the proof we use the so called modified Gauss decompositions[2] which allow to overcome the main disadvantage of any standard Gauss decomposition that not any element of $G$ possesses this decomposition. Namely, if an element $a \in G$ does not admit the Gauss decomposition of some form, then subjecting $a$ to a left shift in $G$ we can get an element having this decomposition.

Let us give examples of multidimensional Toda type equations, namely those corresponding to the cases $l_- = l_+ = l = 1, 2$. For $l = 1$ one has

$$[c_{\pm i}, \gamma^{\pm 1} \partial_{\pm j} \gamma^{\mp 1}] - [c_{\pm j}, \gamma^{\pm 1} \partial_{\pm i} \gamma^{\mp 1}] = 0, \tag{1}$$

$$\partial_{+j} (\gamma^{-1} \partial_{-i} \gamma) = [c_{-i}, \gamma^{-1} c_{+j} \gamma]. \tag{2}$$

For $l = 2$ with the renotation $\upsilon_{\pm i, \pm 1} \equiv \upsilon_{\pm i}$ one has

$$[c_{\pm i}, \upsilon_{\pm j}] = [c_{\pm j}, \upsilon_{\pm i}],$$

$$\partial_{\pm i} \upsilon_{\pm j} \pm [\gamma^{\pm 1} \partial_{\pm i} \gamma^{\mp 1}, \upsilon_{\pm j}] = \partial_{\pm j} \upsilon_{\pm i} \pm [\gamma^{\pm 1} \partial_{\pm j} \gamma^{\mp 1}, \upsilon_{\pm i}],$$

$$[c_{\pm i}, \gamma^{\pm 1} \partial_{\pm j} \gamma^{\mp 1}] - [c_{\pm j}, \gamma^{\pm 1} \partial_{\pm i} \gamma^{\mp 1}] + [\upsilon_{\pm i}, \upsilon_{\pm j}] = 0;$$

$$\partial_{\pm i} \upsilon_{\mp j} = [c_{\mp j}, \gamma^{\mp 1} \upsilon_{\pm i} \gamma^{\pm 1}],$$

$$\partial_{+j} (\gamma^{-1} \partial_{-i} \gamma) = [\upsilon_{-i}, \gamma^{-1} \upsilon_{+j} \gamma] + [c_{-i}, \gamma^{-1} c_{+j} \gamma].$$

**3.** Describe briefly the procedure for obtaining the general solution of the multidimensional Toda type equations. We start with mappings $\gamma_\pm$ taking values in $G_0$ and mappings $\lambda_{\pm i,m}$ with values in $\mathcal{G}_{\pm m}$, which satisfy the conditions $\partial_{\mp i}\gamma_\pm = 0$, $\partial_{\mp i}\lambda_{\pm i,m} = 0$, and the integrability conditions of the equations

$$\mu_\pm^{-1}\partial_{\pm i}\mu_\pm = \sum_{m=1}^{l_{\pm i}-1} \lambda_{\pm i,m} + \gamma_\pm c_{\pm i}\gamma_\pm^{-1}, \tag{3}$$

where $\mu_\pm$ are mappings from $M$ to $G$, such that $\partial_{\pm i}\mu_\mp = 0$. The solution of equations (3) is determined by the initial conditions $\mu_\pm(p) = a_\pm$, where $p$ is some fixed point of $M$ and $a_\pm$ are some fixed elements of $G$. It is clear that the mappings $\mu_\pm$ take values in the subsets $a_\pm N_\pm$ with $N_\pm$ being the Lie subgroups of $G$ corresponding to the Lie subalgebras $\mathcal{N}_\pm = \oplus_{m>0}\mathcal{G}_{\pm m}$. Further, the Gauss decomposition for $\mu_+^{-1}\mu_-$ of the form $\mu_+^{-1}\mu_- = \nu_-\eta\nu_+^{-1}$ gives the mappings $\eta$ and $\nu_\pm$ which take values in $G_0$ and $N_\pm$, respectively. Finally, using the formula $\gamma = \gamma_+^{-1}\eta\gamma_-$ and the decompositions

$$\sum_{m=1}^{l_{\pm i}} \upsilon_{\pm i,m} = \gamma_\pm^{-1}\eta^{\pm 1}(\nu_\pm^{-1}\partial_{\pm i}\nu_\pm)\eta^{\mp 1}\gamma_\pm,$$

we obtain the mappings $\gamma$ and $\upsilon_{\pm i,m}$ which satisfy the multidimensional Toda type equations. Any solution can be obtained using this procedure. In general, different sets of mappings $\gamma_\pm$, $\lambda_{\pm i,m}$, as well as different choices of initial conditions for $\mu_\pm$, can give the same solution. Note that almost all solutions of the multidimensional Toda type equations can be obtained using the mappings $\mu_\pm$ taking values in the subgroups $N_\pm$.

**4.** As an illustration of our general construction consider a particular case of system (1), (2) corresponding to the loop group $\mathcal{L}(GL(2m,\mathbf{C}))$. With an appropriate $\mathbf{Z}$–gradation of the Lie algebra $\mathcal{L}(\mathcal{GL}(2m,\mathbf{C}))$ one has

$$c_{-i} = \begin{pmatrix} 0 & \zeta^{-1}K_{-i} \\ L_{-i} & 0 \end{pmatrix}, \quad c_{+i} = \begin{pmatrix} 0 & K_{+i} \\ \zeta L_{+i} & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.$$

Here each entry in the matrices is an $m\times m$ block, $\beta_{1,2}$ take values in $GL(m,\mathbf{C})$, $\zeta$ is the loop parameter. By this example we show that subsystem (1) gives two sets of generalised wave equations over the coordinates $z^{+i}$ and $z^{-i}$, respectively, while subsystem (2) is a dynamical one and provides a nontrivial mixing of the dependence on the whole set of the coordinates.

If $L_{\pm i} = K_{\pm i}^t$, the system under consideration admits the reduction to the case when $\beta_{1,2}$ take values in the complex orthogonal group $O(m,\mathbf{C})$, which

leads to the equations

$$\partial_{-i}(\beta_1 K_{-j}\beta_2^{-1}) = \partial_{-j}(\beta_1 K_{-i}\beta_2^{-1}), \tag{4}$$
$$\partial_{+i}(\beta_1^{-1} K_{+j}\beta_2) = \partial_{+j}(\beta_1^{-1} K_{+i}\beta_2), \tag{5}$$
$$\partial_{+j}(\beta_1^{-1}\partial_{-i}\beta_1) = K_{-i}\beta_2^{-1}K_{+j}^t\beta_1 - \beta_1^{-1}K_{+j}\beta_2 K_{-i}^t, \tag{6}$$
$$\partial_{+j}(\beta_2^{-1}\partial_{-i}\beta_2) = K_{-i}^t\beta_1^{-1}K_{+j}\beta_2 - \beta_2^{-1}K_{+j}^t\beta_1 K_{-i}. \tag{7}$$

For $K_{\pm i}^t = K_{\pm i}$ system (4)–(7) admits further reduction to the case $\beta_1 = \beta_2 = \beta$, and one ends up with the Cecotti–Vafa equations[5],

$$\partial_{-i}(\beta K_{-j}\beta^{-1}) = \partial_{-j}(\beta K_{-i}\beta^{-1}), \tag{8}$$
$$\partial_{+i}(\beta^{-1} K_{+j}\beta) = \partial_{+j}(\beta^{-1} K_{+i}\beta), \tag{9}$$
$$\partial_{+j}(\beta^{-1}\partial_{-i}\beta) = [K_{-i}, \beta^{-1}K_{+j}\beta]. \tag{10}$$

Equations (8), (9) with $(K_{\pm i})_{kl} = \delta_{ik}\delta_{il}$ are connected with some well known equations in differential geometry[6]. With the same choice of $K_{\pm i}$, a similar connection takes place for more general equations (4), (5). Namely, introduce the notation

$$b_{ij} = \frac{1}{(\beta_1)_{mi}}\partial_{+i}(\beta_1)_{mj}, \qquad i \neq j.$$

It is not difficult to show that equations (5) are equivalent to the relations

$$\beta_1^{-1}\partial_i\beta_1 = K_i b - b^t K_i, \qquad \beta_2^{-1}\partial_i\beta_2 = K_i b^t - bK_i. \tag{11}$$

Here and in what follows for the sake of brevity we omit signs $\pm$ in the indices. The functions $b_{ij}$ defined in such a way satisfy the equations

$$\partial_i b_{ji} + \partial_j b_{ij} + \sum_{k\neq i,j} b_{ik}b_{jk} = 0, \ i \neq j; \tag{12}$$
$$\partial_k b_{ji} = b_{jk}b_{ki}, \ i \neq j \neq k; \tag{13}$$
$$\partial_i b_{ij} + \partial_j b_{ji} + \sum_{k\neq i,j} b_{ki}b_{kj} = 0, \ i \neq j, \tag{14}$$

which are nothing but the zero curvature condition for the connections with components $\beta_{1,2}^{-1}\partial_i\beta_{1,2}$ given by (11). From the other hand, if we have a solution of equations (12)–(14), then integrating equations (11) we come to the mappings $\beta_{1,2}$ which satisfy equations (5). When $\beta_1 = \beta_2$, it appears that the Egorov property, $b_{ij} = b_{ji}$, is valid, and equations (14) are reduced to

$$\sum_{k=1}^{m} \partial_k b_{ij} = 0.$$

Equations (12)–(14) are related to some classical problems of differential geometry, moreover, they are completely integrable[8].

The system arising from equations (6), (7) with the specialisation chosen above, sometimes is called a multidimensional generalisation of the sine–Gordon equation, while those from (14) — generalised wave equations. In particular, for the simplest case with $m = 2$, the integration problem of these equations can be reduced to the solution of two independent sine–Gordon equations.

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